TENTH WORLD CONGRESS ON THE THEORY OF MACHINES AND MECHANISMS
Oulu, Finland, June 20-24, 1999

ON THE GEOMETRY OF APPROXIMATE BI-INVARIANT PROJECTIVE DISPLACEMENT METRICS

Pierre M. Larochelle
Robotics and Spatial Systems Laboratory
Mechanical Engineering Program
Florida Institute of Technology
Melbourne, FL 32901-6975

Keywords: displacement metrics, rigid-body motion, spatial displacements

1 Introduction

This paper presents the underlying geometry of the approximate bi-invariant metrics of Larochelle and McCarthy 1995 and Etzel and McCarthy 1996. The geometry that they algebraically exploit to yield approximate bi-invariant metrics for measuring the distance between two planar locations(Larochelle and McCarthy) or two spatial locations(Etzel and McCarthy) of a finite rigid body is presented. Moreover, we discuss the relationship between the geometry and approximate bi-invariance of these metrics. Finally, the previous discussion serves as a foundation for a geometry based examination of the merits and limitations of these approximately bi-invariant displacement metrics.

Simply stated a metric measures the distance between two points in a set. There exist numerous useful metrics for defining the distance between two points in Euclidean space, however, defining similar metrics for determining the distance between two locations of a finite rigid body is still an area of ongoing research, see Kazerounian and Rastegar 1992, Bobrow and Park 1995, Park 1995, Martinez and Duffy 1995, Larochelle and McCarthy 1995, Etzel and McCarthy 1996, Gupta 1997, Larochelle and Tse 1998, and Chirikjian 1998. In the cases of two locations of a finite rigid body in either $\mathbf{E}^3$(spatial locations) or $\mathbf{E}^2$(planar locations) any metric used to measure the distance between the locations yields a result which depends upon the chosen reference frames, see Bobrow and Park 1995 and Martinez and Duffy 1995. Recall that a bi-invariant metric is independent of choice of both the fixed and moving reference frames. Interestingly, for the specific case

of orienting a finite rigid body in $\mathbf{E}^3$ bi-invariant metrics do exist. For example, in 1983 Ravani and Roth 1983 defined the distance between two orientations of a rigid body as the magnitude of the difference between the associated quaternions and a proof that this metric is bi-invariant may be found in Larochelle and McCarthy 1995.

In 1995 Larochelle and McCarthy presented an algorithm for approximating a set of n locations in planar Euclidean space ($\mathbf{E}^2$) with n spherical orientations in ($\mathbf{E}^3$). By utilizing the bi-invariant metric of Ravani and Roth on spherical orientations they arrived at an approximate bi-invariant metric for planar locations in which the error induced by the spherical approximation is of the order $\frac{1}{\mathsf{R}^2}$, where R is the radius of the approximating sphere(hence the term *projective displacement metric*). Their algorithm for an approximately bi-invariant metric is based upon an algebraic formulation which utilizes Taylor series expansions of *sine*() and *cosine*() terms in homogeneous transforms, see McCarthy 1983.

In 1996 Etzel and McCarthy extended the works of Larochelle and McCarthy by using orientations in $\mathbf{E}^4$ to approximate locations in $\mathbf{E}^3$. Their algorithm for an approximately bi-invariant metric is also based upon an algebraic formulation which utilizes Taylor series expansions of *sine*() and *cosine*() terms, see Ge 1994. Moreover, the error induced by the hyperspherical approximation is also of the order $\frac{1}{\mathsf{R}^2}$, where here R is the radius of the approximating hypersphere.

In this work we present studies of the approximate bi-invariant projective metrics to illuminate the relationships between the geometry and the approximate bi-invariance of these metrics. Moreover, the discussion serves as a foundation for a geometry based examination of the merits and limitations of these approximately bi-invariant displacement metrics.

2 Planar Projective Displacement Metric

We now review how spherical displacements may be used to approximate planar displacements with some finite error associated with the radius R of the sphere, see Larochelle and McCarthy 1995.

Recall that a general planar displacement (a, b, ψ) in the $z = \mathsf{R}$ plane may be expressed as a homogeneous coordinate transformation,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = [A_p] \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & a \\ \sin\psi & \cos\psi & b \\ 0 & 0 & \mathsf{R} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{1}$$

Now consider a general spherical displacement in which the parameters used to describe the displacement are the three angles longitude(θ), latitude(ϕ), and roll(ψ). Using these parameters a general spherical displacement may be written as,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = [A_s] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = Rot(y, \theta) Rot(x, -\phi) Rot(z, \psi) \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2}$$

We now define $\hat{a} = \mathsf{R}\theta$ as the longitudinal arc length and $\hat{b} = \mathsf{R}\phi$ as the latitudinal arc length. Now, if we consider only displacements in the $z = \mathsf{R}$ plane and we expand the trigonometric functions *sine*() and *cosine*() using a Taylor series about 0 and substitute the angles θ and ϕ from above into the expansions then we may we rewrite Eq. 2 as,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & \hat{a} \\ \sin\psi & \cos\psi & \hat{b} \\ 0 & 0 & \mathsf{R} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \frac{1}{\mathsf{R}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\hat{a}\cos\psi - \hat{b}\sin\psi & \hat{a}\sin\psi - \hat{b}\cos\psi & -\frac{1}{2}\left(\hat{a}^2 + \hat{b}^2\right) \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + O(\frac{1}{\mathsf{R}^2}) \tag{3}$$

Note that the first term of Eq. 3 is identical to the equation for general planar displacements, Eq. 1. From the derivation and analysis of Eq. 3 we conclude that spherical displacements may be used to approximate planar displacements with some finite error which is associated with the radius of the sphere. The procedure used to approximate a planar displacement (a, b, ψ) with a displacement on a sphere of radius R is as follows. Using the definition of the arc lengths and the radius of the sphere we obtain the three angles; θ, ϕ, and ψ, which describe the spherical displacement on the sphere of radius R that approximates the prescribed planar displacement: $\theta = \frac{a}{\mathsf{R}}$, $\phi = \frac{b}{\mathsf{R}}$, and, $\psi = \psi$. Finally, let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the quaternion representations of two spherical displacements $(\theta, \phi, \psi)_1$ & $(\theta, \phi, \psi)_2$ which approximate two planar displacements $(a, b, \psi)_1$ & $(a, b, \psi)_2$. The measure d of the magnitude of the relative displacement between the two planar displacements is defined as,

$$d^2 = (\mathbf{q}_1 - \mathbf{q}_2)^T(\mathbf{q}_1 - \mathbf{q}_2). \tag{4}$$

3 Spatial Projective Displacement Metric

Etzel and McCarthy extended the above planar methodology to spatial displacements by using orientations in $\mathbf{E}^4$ to approximate locations in $\mathbf{E}^3$. They showed that a 4x4 homogeneous transform can be approximated by a pure rotation $[D]$ in $\mathbf{E}^4$,

$$[D] = [J(\alpha, \beta, \gamma)][K(\theta, \phi, \psi)] \tag{5}$$

where,

$$J(\alpha, \beta, \gamma)] = \begin{bmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ -\sin\beta\sin\alpha & \cos\beta & 0 & \sin\beta\cos\alpha \\ -\sin\gamma\cos\beta\sin\alpha & -\sin\gamma\sin\beta & \cos\gamma & \sin\gamma\cos\beta\cos\alpha \\ -\cos\gamma\cos\beta\sin\alpha & -\sin\beta\cos\gamma & -\sin\gamma & \cos\gamma\cos\beta\cos\alpha \end{bmatrix}$$

and,

$$K(\theta,\phi,\psi)] = \begin{bmatrix} & & & 0 \\ & [A_s] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The angles α, β and γ are defined as follows: $\tan(\alpha) = \frac{d_x}{\mathsf{R}}$, $\tan(\beta) = \frac{d_y}{\mathsf{R}}$, and $\tan(\gamma) = \frac{d_z}{\mathsf{R}}$ where d_x, d_y, and d_z are the components of the translation vector $\mathbf{d}$ of the displacement and R is the radius of the hypersphere. Finally, let $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ be the biquaternion representations of two hyperspherical displacements $(\theta,\phi,\psi,\alpha,\beta,\gamma)_1$ & $(\theta,\phi,\psi,\alpha,\beta,\gamma)_2$ which approximate two spatial displacements $(d_x,d_y,d_z,\theta,\phi,\psi)_1$ & $(d_x,d_y,d_z,\theta,\phi,\psi)_2$. The measure d of the magnitude of the relative displacement between the two spatial displacements is defined as,

$$d^2 = (\hat{\mathbf{q}}_1 - \hat{\mathbf{q}}_2)^T(\hat{\mathbf{q}}_1 - \hat{\mathbf{q}}_2). \tag{6}$$

4 Projective Metric Geometry

Here we analyze the geometry of the projective displacement metrics. In this analysis we seek to examine the role of the radius of the approximating sphere with respect to the approximate bi-invariance of the metric. The goal is to determine criteria for selecting appropriate radii for a set of prescribed displacements. Our focus will be the planar metric but the results apply equally well to the general spatial case.

Etzel and McCarthy present a numerical example and conclude that "Decreasing R increases the influence of the translation terms". They go on to state "The conclusion is the parameter R is a physical realization of the weighting term often used to construct metrics combining rotations and translations in space (Park 1995)". Here, we perform a more formal study of the effect of R and its role as a weighting term.

Consider $\lim_{\mathsf{R}\to\infty}$ of the spherical approximation to a planar displacement. Eq. 3 then becomes,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & \infty \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + [0]. \tag{7}$$

Note that the region of approximation degenerates to the point $(0,0,\infty)$ and that the planar displacement is now approximated as a pure planar rotation. Since the displacement is purely rotational the metric is fully bi-invariant.

Consider now $\lim_{\mathsf{R}\to 0}$ of the spherical approximation to a planar displacement. Eq. 3 then becomes,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & \pm\infty \\ \sin\psi & \cos\psi & \pm\infty \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pm\infty & \pm\infty & -\infty \end{bmatrix}. \tag{8}$$

Note that the region of approximation degenerates to the infinity plane $z = 0$ and that infinite translation dominates the displacement. This concurs with and supports the observations of Etzel and McCarthy. Moreover, since the displacement is purely translational the projective metric is again bi-invariant.

We have shown that as $\lim_{R\to\infty}$ the projective metric disregards the translation terms of planar displacements and that as $\lim_{R\to 0}$ the projective metric disregards the rotation terms. Hence the interpretation of R as a physical realization of the weighting term often used to construct metrics combining rotations and translations in space is valid. We proceed now by seeking criteria to determine appropriate radii for a set of prescribed displacements.

First, we present a numerical study to illustrate the conclusions of the previous section. In Fig. 1 we consider the magnitude of two planar displacements: $(0, 0, 60)$ and $(1, 1, 0)$. Note that the magnitude of the purely rotational displacement is independent of the choice of R while the magnitude of the purely translational displacement has a 99% variation with respect to R.

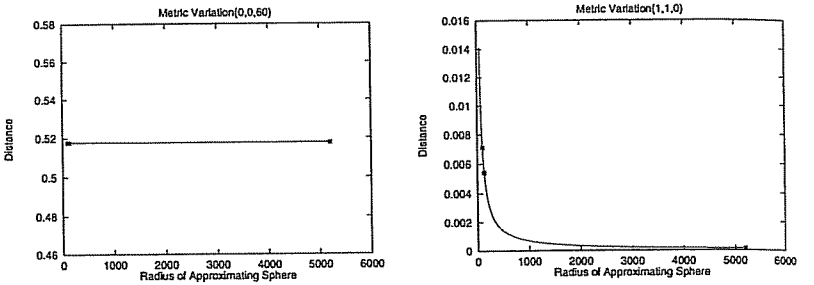

Figure 1: Metric Variation for Uncoupled Displacements

For a prescribed set of planar displacements Larochelle and McCarthy suggest $R = \frac{24L}{\pi} \approx 7.6L$ where L is the maximum translation component of the workspace. In related works Etzel and McCarthy select $R = 10L$ and Tse and Larochelle select $R = 400L$. Here we examine the magnitude of two of the displacements studied in Larochelle and McCarthy for $50 \leq R \leq 400L$, see Fig. 2 where $*$ indicates the values of R corresponding to the previous works. As expected, the displacement with the larger translation is more sensitive to choice of R. The displacement $(-1.5, 3, 20)$ has a 1.5% variation in magnitude with respect to R while the $(13, 12.5, -35)$ displacement exhibits a 9.8% variation.

5 Conclusions

We have shown that the interpretation of the projective sphere radius R is a physical realization of the weighting term often used to construct metrics combining rotations and translations in space is valid. Moreover, as $\lim_{R\to\infty}$ the projective metric disregards the

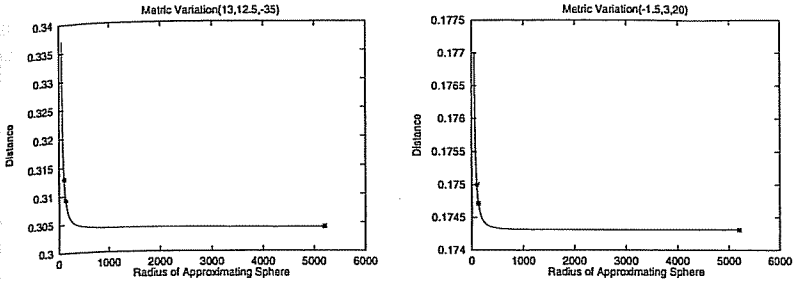

Figure 2: Metric Variation for Coupled Displacements

translation terms of displacements and that as $\lim_{R\to 0}$ the projective metric disregards the rotation terms. Finally, we have illustrated that the radius selection guidelines presented by Larochelle and McCarthy($R = \frac{24L}{\pi}$) and Etzel and McCarthy($R = 10L$) yield acceptable weighting of rotations and translations.

6 Acknowledgments

The support of the National Science Foundation, grants #DMI-9612062 and #DMI-9816611, is gratefully acknowledged.

References

[1] P. M. Larochelle and J. M. McCarthy. Planar motion synthesis using an approximate bi-invariant metric. *ASME J. of Mechanical Design*, 117(3):646–651, 1995.

[2] K.R. Etzel and J. M. McCarthy. A metric for spatial displacements using biquaternions on so(4). In *Proc. IEEE Robotics and Automation Conf.*, volume 4, pages 3185–3190, 1996.

[3] K. Kazerounian and J. Rastegar. Object norms: A class of coordinate and metric independent norms for displacements. In *Proc. ASME DETC Mechanisms Conf.*, pages 271–275, Phoenix, Arizona, September 1992.

[4] J.E. Bobrow and F.C. Park. On computing exact gradients for rigid body guidance using screw parameters. In *Proc. ASME Design Engineering Technical Confs.*, volume 1, pages 839–844, Boston, Massachusetts, 1995.

[5] F.C. Park. Distance metrics on the rigid-body motions with applications to mechanism design. *ASME J. of Mechanical Design*, 117(1), March 1995.

[6] J.M.R. Martinez and J. Duffy. On the metrics of rigid body displacements for infinite and finite bodies. *ASME J. of Mechanical Design*, 117:41–47, March 1995.

[7] K. C. Gupta. Measures of positional error for a rigid body. *ASME J. of Mechanical Design*, 119:346–348, September 1997.

[8] P. M. Larochelle and D. Tse. A new method of task specification for spherical mechanism design. In *Proc. ASME DETC Mechanisms Conf.*, Atlanta, Georgia, August 1998.

[9] G.S. Chirikjian. Convolution metrics for rigid body motion. In *Proc. ASME DETC Mechanisms Conf.*, Atlanta, Georgia, August 1998.

[10] B. Ravani and B. Roth. Motion synthesis using kinematics mappings. *ASME J. of Mech., Trans., and Auto. in Des.*, 105:460–467, September 1983.

[11] J. M. McCarthy. Planar and spatial rigid body motion as special cases of spherical and 3-spherical motion. *ASME J. of Mech., Trans., and Auto. in Des.*, 105:569–575, September 1983.

[12] Q. J. Ge. On the matrix algebra realization of the theory of biquaternions. In *Proc. ASME DETC Mechanisms Conf.*, volume 70, pages 425–432, Minneapolis, Minnesota, September 1994.